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Sea Spectra Revisited

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This paper comprises a compendium of updated spectrum formulations, Rayleigh factors, and associated wave height and period relationships—prepared for easy understanding and application.

1. Introduction

THE BASIC spectrum formulations that have been developed through the years, and which have found acceptance for general application, are given here in their most commonly used format. These spectra range from the "fully developed sea," having only one parameter in its formulation, to double-peaked spectra representing combined sea and swell which utilize six parameters. In between are those of two to five parameters, more generally applicable to strong seas having sharply peaked spectral shape, which is of primary interest in design assessments. These latter are covered in some detail, and a general comparison is made on the basis of a typical recorded North Sea spectrum.

Off and on, there have been variations in the selection of the most representative period parameter. This is reviewed in some detail and the proper values of the various period designations are shown, along with their general relationship to each other (and to the peak frequency, ω_0 or f_0, which is now the favored parameter and logically so, as will be seen). Similarly, a listing of the various spectrum ordinate scales that have been used is shown, and the resulting areas under the curve are compared with the presently accepted statistical variance σ^2 of the wave record.

Wave height factors are shown as derived from the Rayleigh distribution. A clear distinction is seen between "the most probable highest" wave and "the average highest" wave, with the latter being significantly higher (and almost identical to the root mean square of the highest waves). The same trend is observed for modified Rayleigh distributions due to breaking waves which have lower height probabilities.

Wind generation of waves is formulated herein, based on an array of wave heights and periods observed over a number of years. The maximum height to period ratio obtained is in general accord with design requirements of cognizant agencies, and the overall generating and decay ranges appear consistent with more recent North Sea measurements.

Finally, for those who have had little or no prior exposure to spectrum theory, an appendix is included that offers a short introduction, adapted from this author's 1967 paper "Sea Spectra Simplified." It would be of benefit to review this section first before addressing the above subject matter.

It is the purpose of this paper to present basic spectrum formulations that are current and well-established, along with associated factors in a clear and easy-to-grasp manner, for the benefit of those not fully acquainted with the details and others who may have forgotten them. Attention has not been given here to various corrective influences, such as wave grouping, mass transport, the effect of current, and height/period dependency, all of which require further research and data corroboration to achieve consistency and straightforward methodology. In the meantime, those who are cognizant of the basic fundamentals as set forth should be well prepared to cope with such extensions of the state-of-the-art, if and when the need arises.

2. The standard sea spectrum function

2.1 The basic spectrum function

The theoretical function that is most widely recognized and referred to throughout the world, to represent the state of the sea in the form of an energy spectrum, is the two-parameter spectrum developed by Bretschneider in 1959. (See Fig. 1.)

In the early development, the wave period relationship was emphasized. In its simplest form, this may be expressed as:

$$S_{(T)} = \alpha T^3 e^{-\beta T^4}$$

The two general parameters, α and β, are readily replaced by two meaningful functions that precisely define the spectrum in terms of total energy and peak energy. Thus:

$$S_{(T)} = \frac{3E}{T_P{}^4} T^3 e^{-\frac{3}{4}\left(\frac{T}{T_P}\right)^4} \qquad (1)$$

where E is the total area (representative of total energy) under the spectrum curve, integrated from 0 to ∞, and T_P is the wave period at which the spectral energy, $S_{(T)}$, has its peak or maximum value.

To transform this period spectrum into the more popular frequency spectra (Fig. 2), the energy of each differential strip should be the same, and thus we have the following equalities:

$$T = \frac{2\pi}{\omega} = \frac{1}{f}$$

$$S_{(T)}\, dT = -S_{(\omega)}\, d\omega = -S_{(f)}\, df$$

Then, the spectrum of the circular frequency ω is:

$$S_{(\omega)} = \frac{T^2}{2\pi} S_{(T)} = 3E \left(\frac{2\pi}{T_P}\right)^4 \omega^{-5} e^{-\frac{3}{4}\left(\frac{2\pi}{T_P}\right)^4 \omega^{-4}}$$

and that of the cyclic frequency f is:

$$S_{(f)} = T^2 S_{(T)} = \frac{3E}{T_P{}^4} f^{-5} e^{-\frac{3}{4 T_P{}^4} f^{-4}}$$

The frequency at which these spectra have their peak or maximum values can be shown to be:

$$\omega_0 = \left(\frac{3}{5}\right)^{\frac{1}{4}} \cdot \frac{2\pi}{T_p}$$

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Fig. 1 Characteristic period spectrum

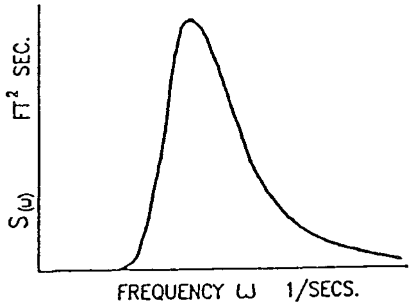

Fig. 2 Characteristic frequency spectrum

$$f_0 = \left(\frac{3}{5}\right)^{\frac{1}{4}} \cdot \frac{1}{T_p}$$

or clearly the peak frequency is not the reciprocal value of the peak period. (The subscript notation is changed to further emphasize the difference.)

Substituting, the frequency spectra are given in their simplest form:

$$S_{(\omega)} = 5E\,\omega_0^{\,4}\,\omega^{-5}\,e^{-\frac{5}{4}\left(\frac{\omega_0}{\omega}\right)^4} \quad (2)$$

$$S_{(f)} = 5E\,f_0^{\,4}\,f^{-5}\,e^{-\frac{5}{4}\left(\frac{f_0}{f}\right)^4} \quad (3)$$

2.2 Period relationships of spectrum

It has been customary to replace the peak period and peak frequencies in the previous spectrum equations with other representative periods that may be more visually apparent in the seaway. A number of such periods have been in use, albeit frequently undefined in presentation. Following are most of the common ones, defined and evaluated:

T_0 The "optimum" or "modal" period, used with the frequency spectrum to denote the reciprocal of the peak frequency.

$$T_0 = \frac{2\pi}{\omega_0} = \frac{1}{f_0} = \left(\frac{5}{3}\right)^{\frac{1}{4}} T_p$$

T_a The true average period of the elemental waves in the spectrum. Used in the period spectra, it is determined as:

$$T_a = \frac{\int_0^\infty T\,S_{(T)}dT}{\int_0^\infty S_{(T)}dT} = \left(\frac{4}{3}\right)^{\frac{1}{4}} \Gamma\left(\frac{5}{4}\right) T_P = \left(\frac{4}{5}\right)^{\frac{1}{4}} \Gamma\left(\frac{5}{4}\right) T_0$$

T_1 T_v The reciprocal of the true average frequency determined as:

$$T_1 = \frac{2\pi \int_0^\infty S_{(\omega)}\,d\omega}{\int_0^\infty \omega\,S_{(\omega)}\,d\omega} = \frac{T_0}{\left(\frac{5}{4}\right)^{\frac{1}{4}} \Gamma\left(\frac{3}{4}\right)}$$

T_1 is the period adopted by ITTC in 1969 as their standard reference, labeled the "characteristic period." Earlier, ISSC suggested that this might be equivalent to the period estimated from shipboard observations, which they called T_v.

T_z The average period of zero upcrossings in the wave record, which is the period most readily determined from the wave record and model test data. From the frequency spectrum:

$$T_z = 2\pi \left[\frac{\int_0^\infty S_{(\omega)}\,d_\omega}{\int_0^\infty \omega^2 S_{(\omega)}\,d_\omega}\right]^{1/2} = \left(\frac{4}{5\pi}\right)^{1/4} T_0$$

(More precisely, this is the reciprocal of 1/2 the average frequency of all zero crossings, up and down.)

T_c The average period of all of the crests in the wave record. It is determined from the frequency spectrum as:

$$T_c = 2\pi \left[\frac{\int_0^\infty \omega^2 S_{(\omega)}\,d\omega}{\int_0^\infty \omega^4 S_{(\omega)}\,d\omega}\right]^{\frac{1}{2}}$$

Again, more precisely, this is the reciprocal of 1/2 the average frequency of crests and troughs. Unfortunately, the function $\int_0^\infty \omega^4 S_{(\omega)} d\omega$ approaches infinity for this theoretical spectrum, and it is necessary to truncate the spectrum by neglecting high frequency components (to correlate with actual wave records which do not show them). Thus, this period should not be used as an index with the full theoretical spectrum.

T_s The "significant period," which is defined as the average period of the 1/3 highest wave in the record. It is determined directly by measurement of the time between significant crests, either by analysis of the recorded data or by stop watch, depending on what is available. Since personal judgement is involved in what are significant crests, the value of the significant period relative to the spectrum is not fully established (nor can it be determined mathematically from the spectrum function). Nevertheless, most early oceanographic data were obtained by these methods and some specifications still refer to them. In various studies, Bretschneider found that the significant period falls between the values of the peak period (T_p) and the peak frequency (T_0), and proposed the value:

$$T_s = \left(\frac{4}{3}\right)^{\frac{1}{4}} T_p = \left(\frac{4}{5}\right)^{\frac{1}{4}} T_0$$

Each of the periods defined previously has been used at one time or another as the designated period parameter of the spectrum equation (with the possible exception of T_c). It is apparent that unless the particular period in use is clearly defined in a presentation, confusion and error can result.

Further, it is revealing to see the relative positions of the various periods in the spectrum, as shown in Fig. 3. Clearly, for the frequency spectrum shown, the modal period (peak frequency) is dominant and explicit. The other periods will vary in relative value, depending on whether the spec-

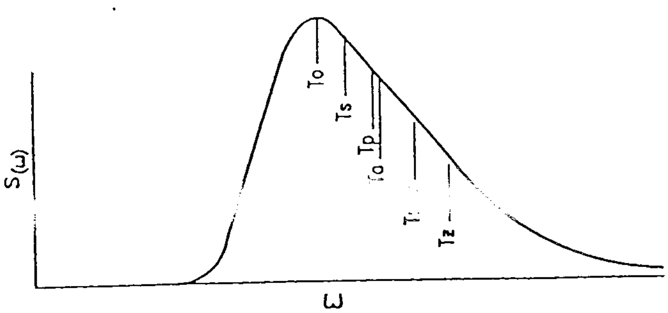

Fig. 3 Frequency spectrum showing position of various periods

trum is of this standard theoretical form, some other formulation, or an actual wave record.

For this spectrum:

$$T_s = \left(\frac{4}{5}\right)^{\frac{1}{4}} T_0 = 0.946\, T_0$$

$$T_p = \left(\frac{3}{5}\right)^{\frac{1}{4}} T_0 = 0.880\, T_0$$

$$T_a = \left(\frac{4}{5}\right)^{\frac{1}{4}} \Gamma\left(\frac{5}{4}\right) T_0 = 0.857\, T_0$$

$$T_z = \left(\frac{4}{5\pi}\right)^{\frac{1}{4}} T_0 = 0.710\, T_0$$

$$T_1 = \frac{T_0}{\left(\frac{5}{4}\right)^{\frac{1}{4}} \Gamma\left(\frac{3}{4}\right)} = 0.772\, T_0$$

The modal period T_0 therefore is proposed as the standard period parameter to be used in the frequency spectrum equation, (and the peak period T_p when using the period spectrum) to which all other period information should be referred.

The zero-upcrossing period, T_z, remains a popular second choice, more because of its direct evaluation from the wave record rather than any importance as a key indicator of the spectrum characteristics (it becomes important in determining the number of waves generated in a given time).

2.3 Area under spectrum curve and its relation to significant wave height

It is seen from the previous basic spectrum relationships that the area "E" under the curve is directly proportional to "$S_{(\omega)}$," the ordinate of spectral density. Any change in the ordinate scale produces the same change in area, without affecting the characteristic shape or the various period or frequency relationships.

It is generally considered that the Rayleigh distribution of wave heights applies to the spectrum, and thus the area of the spectrum is directly proportional to the square of the significant height. This is shown by noting that the mean square height of the wave record represents the same energy as the sum of the heights of all of the small regular waves of the spectrum:

$$\overline{H}^2 = (h_1^2 + h_2^2 + h_3^2 + \ldots\ldots)$$

and since from the Rayleigh distribution, $H_S^2 = 2\,\overline{H}^2$ we get:

$$H_S^2 = 16\Sigma \tfrac{1}{2}\, a^2 = 8\Sigma a^2 = 2\Sigma h^2 = \Sigma 2h^2$$

Note that the significant height, H_s, is defined as the average of the 1/3 highest waves in the spectrum, and is used as the standard reference of wave height characteristics. See Section 6 for details.

Thus, depending on the choice of ordinate, as some form of amplitude squared or height squared representation, the area E must be multiplied by a factor suitable to that ordinate, to equate to H_s^2. The following spectral density scales have been used:

$S_{\frac{1}{2}a^2}$ is the spectrum given in terms of 1/2 the square of the amplitude of the elemental waves. This form is used almost exclusively among today's analysts, since the area under the curve is equal to the statistical variance, σ^2, of the wave record. The area is then $E = \sigma^2 = H_s^2/16$.

$S_a{}^2$ is the spectrum in terms of amplitude squared. This is the form originally used by St. Denis and Pierson and others following. Its area $E = H_s^2/8$.

$S_h{}^2$ is the spectrum in terms of the square of the height of the elemental waves, which has been used by Bretschneider. In this case, the area under the curve is equal to the mean square height of the wave record, or $E = \overline{H}^2 = H_s^2/2$.

$S_{2h}{}^2$ is the spectrum in terms of twice the square of the elemental heights. It has no particular merit other than that the area $E = H_s^2$ directly.

Notation for the ordinate scale used has not always been made clear. One can avoid guesswork by simple reviewing the constants of a given spectrum equation. Thus, if we have (for this standard two parameter spectrum)

$$S_{?(\omega)} = k_1 H_s^2\, \omega^{-5} e^{-k_2 \omega^{-4}}$$

Then, it follows that:

$$E = \frac{k_1}{4k_2} H_s^2$$

2.4 Preferred reference form of spectrum equation

With its preferred analytical form and its widespread usage, the standard spectrum today is that using $a^2/2$ as its ordinate. With $H_s^2/16$ representing total energy, E, equations (2) and (3) become:

$$S_{(\omega)} = \frac{5}{16} H_s^2\, \omega_0^4\, \omega^{-5} e^{-\frac{5}{4}\left(\frac{\omega_0}{\omega}\right)^4} \quad (4)$$

$$S_{(f)} = \frac{5}{16} H_s^2 f_0^4 f^{-5} e^{\frac{-5}{4}\left(\frac{f_0}{f}\right)^4} \quad (5)$$

Optionally, if a period designation is preferred, the following may be substituted in the above:

$$T_0 = \frac{2\pi}{\omega_0} = \frac{1}{f_0} = \left(\frac{5\pi}{4}\right)^{\frac{1}{4}} T_z$$

It is common practice now to use the notation $S_{(\omega)}$ or $S_{(f)}$, without the qualifying subnotation ½ a^2 being written. It is so used throughout this paper as well.

2.5 ISSC spectrum and the ITTC spectrum

In 1964, the International Ship Structure Committee presented a spectrum based on visual determination of the average height and period of the wave system. With the average height noted to be a function of the significant height, the equation could then be expressed as follows:

$$S_{(f)} = 0.11\, H_s^2\, T_v\, (T_v f)^{-5} e^{-0.44\,(T_v f)^{-4}} \quad (6)$$

Now, as given in Section 2.2

$$T_\nu = \frac{T_o}{\left(\frac{5}{4}\right)^{1/4} \Gamma\left(\frac{3}{4}\right)} \text{ or } T_\nu^{-4} = 2.82 f_0^{\,4}$$

from which it is seen that the ISSC spectrum is practically identical to equation (5).

Similarly, in 1978, the International Towing Tank Conference, (ITTC) proposed the equation

$$S_{(\omega)} = 173\, H_s^{\,2}\, T_1^{-4}\, \omega^{-5}\, e^{-691\,(T_1\,\omega)^{-4}} \qquad (7)$$

Again, from Section 2.2

$$T_1 = \frac{T_0}{\left(\frac{5}{4}\right)^{1/4} \Gamma\left(\frac{3}{4}\right)} \text{ or } T_1^{-4} = \omega_0^{\,4}/553$$

and thus the ITTC equation is found equivalent to equation (4).

3. Pierson-Moskowitz and JONSWAP spectra

3.1 Original Pierson-Moskowitz spectrum

Following the format of the earlier work of Neumann, the Pierson-Moskowitz spectrum was first introduced in 1964, and in terms of the wind speed, V, measured at a height of 60 ft above the sea surface:

$$S_{(\omega)} = 0.0081\, g^2 \omega^{-5} e^{-0.74 g^4\,(V\omega)^{-4}} \qquad (8)$$

Some time later, the P–M data were reanalyzed to develop the relationship

$$0.74\, \frac{g^4}{V^4} = \frac{5}{4}\, \omega_0^{\,4}$$

from which the spectrum equation could be expressed:

$$S_{(\omega)} = 0.0081\, g^2\, \omega^{-5}\, e^{-\frac{5}{4}\left(\frac{\omega_0}{\omega}\right)^4} \qquad (8a)$$

or as has frequently been applied:

$$S_{(\omega)} = 0.0081\, g^2\, \omega^{-5}\, e^{-\frac{16\,\pi^3}{T_z^{\,4}}\,\omega^{-4}} \qquad (8b)$$

Note that this is a one-parameter spectrum (ω_0 or T_z in the above) and that

$$T_z = 11.1 \sqrt{\frac{H_s}{g}}$$

The P-M spectrum was the standard reference for some time past, until it was realized that its proper use is restricted to fully developed seas as generated by relatively moderate winds over very large fetches. To account for the more prevalent conditions of high winds over relatively short fetches, which produce spectra covering a range of lower periods for a given wave height, the more general two-parameter formulation is needed. In fact, in such areas as the North Sea, many spectra are so highly peaked in shape as to require multi-parameter treatment.

3.2 Modified Pierson-Moskowitz spectrum

This is the same two parameter spectrum that was devised by Bretschneider, and which preceded even the basic P-M spectrum. It is not clear why various cognizant agencies and analysts have labeled it "The Modified Pierson-Moskowitz Spectrum," which through usage has gained some undeserved acceptance.

In any event, this spectrum has already been covered in detail in Section 2, and there is no need to elaborate further.

3.3 The JONSWAP spectrum

JONSWAP (Joint North Sea Wave Project) was organized in the early 1970s for the purpose of recording North Sea wave patterns on a systematic basis, analyzing the data by spectral methods, and parameterizing the resulting spectra in an equation that would accommodate a range of spectrum shapes, from those sharply peaked to those representing the fully developed P-M limit.

To accomplish the most meaningful results, a number of wave recording stations were positioned along a course from the German coast extending west for about 100 miles (160 km). Then, with a wind coming directly from shore, the various fetch distances and wind speeds would be established at each station, along with the knowledge that the generated waves had no prior history, such as might corrupt the resulting data. With conditions so closely controlled, it was anticipated that the wave spectra would start out sharply peaked (consistent with general observations on early developing seas) and gradually ease towards the fully developed spectrum according to Pierson-Moskowitz. On this basis, the Jonswap Spectrum formulation was developed, as follows:

$$\underbrace{S_{(f)} = a g^2\, (2\pi)^{-4} f^{-5}\, e^{-\frac{5}{4}\left(\frac{f_0}{f}\right)^4}}_{\textit{BASIC}\ \text{P–M}} \cdot \underbrace{\nu^{e^{-\frac{(f-f_0)^2}{2\sigma^2 f_0^{\,2}}}}}_{\textit{PEAK FACTOR}} \qquad (9)$$

In this instance, the cyclic frequency (hertz or f) was used.

Figure 4 shows the significance of each of the factors.

- a = "constant" in the P-M spectrum = 0.0081 but is here allowed to vary to suit the data
- f_0 = peak frequency
- ν = enhancement factor by which the P-M peak energy is multiplied to get the peak energy value of the spectrum
- σ = width factor of the enhanced peak, σ_a being used for frequencies less than f_0, σ_b for those greater than f_0

Reportedly, some 2000 wave records were analyzed in the course of the project, and applied to the evaluation of the spectrum parameters, and the findings then published. This was a noble and inspired undertaking!

The results clearly indicate that sharply peaked spectra are common in the North Sea when waves are being generated; and this has been substantiated by data from other North Sea locations under somewhat more severe conditions than experienced in JONSWAP. However, what came as a surprise was that there was little if any tendency for the spectra to "settle down" to the basic P-M fully developed formulation as the fetch increased (several opinions were expressed that perhaps such fully-developed condition may never occur).

In view of this latter observation, JONSWAP then may have been amended to eliminate the P-M dependency, with a

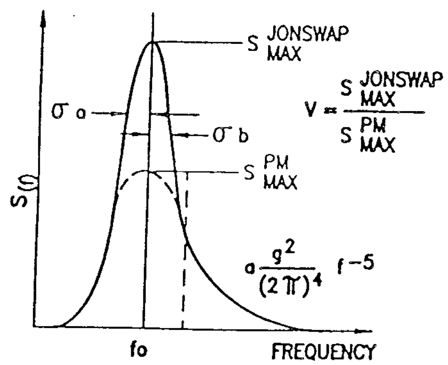

Fig. 4

simpler formulation for easier analysis and evaluation. However, with all of the data having been processed in the existing JONSWAP form, that spectrum function was retained and it continues to be specified for many North Sea operations. In general, where site-specific data have not been defined, the average values obtained in the JONSWAP experiment are considered:

$a = 0.0081$, $\nu = 3.3$ — Along with a selected peak period, $T_0 = 1/f_0$, and which indicates that the significant wave height should be in the order of $H_s = 0.005g\,T_0^2$ for this set of values.
$\sigma_a = 0.07$, $\sigma_b = 0.09$

However, where spectrum data have been developed at a particular site, the value of several of the JONSWAP parameters probably have to be manipulated in order to get a reasonable fit with the data, insofar as the formulation has no analytical basis that would allow ready correction (see Fig. 7).

4. Multi-parameter spectra based on standard function

4.1 Generalized formulations

For sharply peaked spectra, such as those addressed by JONSWAP, more tractable equations can be obtained by use of the standard form.

$$S_{(\omega)} = \alpha\omega^{-\ell}\, e^{-\beta\omega^{-n}}$$

The complete integral includes a gamma function, as follows:

$$\int_0^{\infty} \alpha\, \omega^{-\ell}\, e^{-\beta\omega^{-n}}\, d\omega = \frac{H_s^2}{16} = \frac{\alpha}{n}\frac{\Gamma\left(\frac{\ell-1}{n}\right)}{\beta^{\left(\frac{\ell-1}{n}\right)}} \tag{10}$$

The slope of the spectrum curve is $d\,S(\omega)/d\omega = 0$ at the frequency ω_0, which leads to the value

$$\beta = \frac{\ell}{n}\,\omega_0^{\,n}$$

and thus

$$\alpha = \frac{n\,H_s^2}{16}\left(\frac{\ell}{n}\right)^{\frac{\ell-1}{n}} \omega_0^{\,\ell-1}/\Gamma\left(\frac{\ell-1}{n}\right)$$

The complete equation for the spectrum may then be written:

$$S_{(\omega)} = \frac{n\,H_s^2}{16}\frac{\left(\frac{\ell}{n}\right)^{\frac{\ell-1}{n}}}{\Gamma\left(\frac{\ell-1}{n}\right)}\,\omega_0^{\,\ell-1}\,\omega^{-\ell}\,e^{-\frac{\ell}{n}\left(\frac{\omega_0}{\omega}\right)^n} \tag{11}$$

At the important peak position, ω_0, the spectrum value is:

$$S_{(\omega)_0} = \frac{n\,H_s^2}{16}\frac{\left(\frac{\ell}{n}\right)^{\frac{\ell-1}{n}}}{\Gamma\left(\frac{\ell-1}{n}\right)}\,\omega_0^{-1}\,e^{-\frac{\ell}{n}} \tag{12}$$

The moments of the spectrum defined as

$$m_x = \int_0^{\infty} \omega_x\, S_{(\omega)}\, d\omega$$

where $x = 0, 1, 2 \ldots$ are readily seen to be

$$m_x = \frac{\alpha}{n}\frac{\Gamma\left(\frac{\ell-1-x}{n}\right)}{\beta^{\left(\frac{\ell-1-x}{n}\right)}}$$

From this, various periods of the spectrum can be determined.

The "characteristic" period, T_1 *(or* T_v*)*

$$\frac{T_1}{2\pi} = \frac{m_0}{m_1} = \frac{\Gamma\left(\frac{\ell-1}{n}\right)}{\Gamma\left(\frac{\ell-2}{n}\right)}\left(\frac{\ell}{n}\right)^{-\frac{1}{n}}\omega_0^{-1} \tag{13}$$

The zero-upcrossing period, T_z

$$\left(\frac{T_z}{2\pi}\right)^2 = \frac{m_0}{m_2} = \frac{\Gamma\left(\frac{\ell-1}{n}\right)}{\Gamma\left(\frac{\ell-3}{n}\right)}\left(\frac{\ell}{n}\right)^{-\frac{2}{n}}\omega_0^{-2} \tag{14}$$

The "total crests" period, T_c

$$\left(\frac{T_c}{2\pi}\right)^2 = \frac{m_2}{m_4} = \frac{\Gamma\left(\frac{\ell-3}{n}\right)}{\Gamma\left(\frac{\ell-5}{n}\right)}\left(\frac{\ell}{n}\right)^{-\frac{2}{n}}\omega_0^{-2} \tag{15}$$

It may be noted that when $\ell = 5$ and $n = 4$, all of the above relationships reduce to those of the standard two parameter functions as previously given in Section 2.

4.2 Three-parameter spectra

Keeping the factor $n = 4$ as a constant value, with ℓ being variable, results in the 3-parameter spectrum made popular by Ochi:

$$\text{Ochi 3}\quad S_{(\omega)} = \frac{H_s^2}{4}\left(\frac{\ell}{4}\right)^{\frac{\ell-1}{4}}\frac{\omega_0^{\ell-1}\,\omega^{-\ell}}{\Gamma\left(\frac{\ell-1}{4}\right)}\,e^{-\frac{\ell}{4}\left(\frac{\omega_0}{\omega}\right)^4} \tag{16}$$

$$S_{(\omega_0)} = \frac{H_s^2}{4}\left(\frac{\ell}{4}\right)^{\frac{\ell-1}{4}}\frac{\omega_0^{-1}}{\Gamma\left(\frac{\ell-1}{4}\right)}\,e^{-\frac{\ell}{4}} \tag{17}$$

An alternate 3-parameter spectrum is introduced here, providing a simpler yet effective formulation, by keeping the relationship $\ell = n + 1$:

$$\text{Alt 3}\quad S_{(\omega)} = \frac{H_s^2}{16}(n+1)\,\omega_0^{\,n}\,\omega^{-(n+1)}\,e^{-\frac{n+1}{n}\left(\frac{\omega_0}{\omega}\right)^n} \tag{18}$$

$$S_{(\omega)_0} = \frac{H_s^2}{16}(n+1)\,\omega_0^{-1}\,e^{-\frac{n+1}{n}} \tag{19}$$

A measured sea spectrum whose characteristics have been determined (ω_0, $S_{(\omega)_0}$, H_s) can readily be approximated by either of the above three-parameter spectra, by evaluating the product

$$\frac{\omega_0\, S_{(\omega)_0}}{H_s^2}$$

and finding the appropriate value of ℓ directly, Fig. 5, from which the equation for $S_{(\omega)}$ can be finalized.

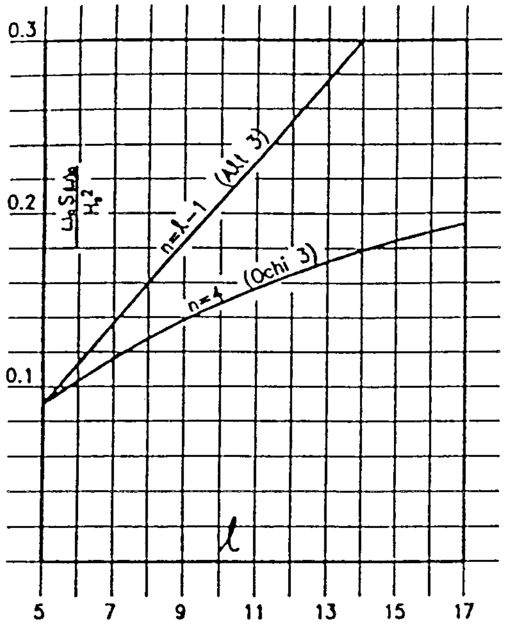

Fig. 5

4.3 Four-parameter spectra

With ℓ and n taken as independent variables in equation (11), a four-parameter spectrum is produced wherein the characteristic T_z is represented along with ω_0, $s_{(\omega)_0}$, and H_s, which are the four major indices of a given spectrum.

With coordinate values of

$$\frac{\omega_0 \, S_{(\omega)_0}}{H_s^2} \quad \text{and} \quad T_z \omega_0,$$

from equations (12) and (14), the appropriate values of ℓ and n may be determined from Fig. 6, and applied to equation (11).

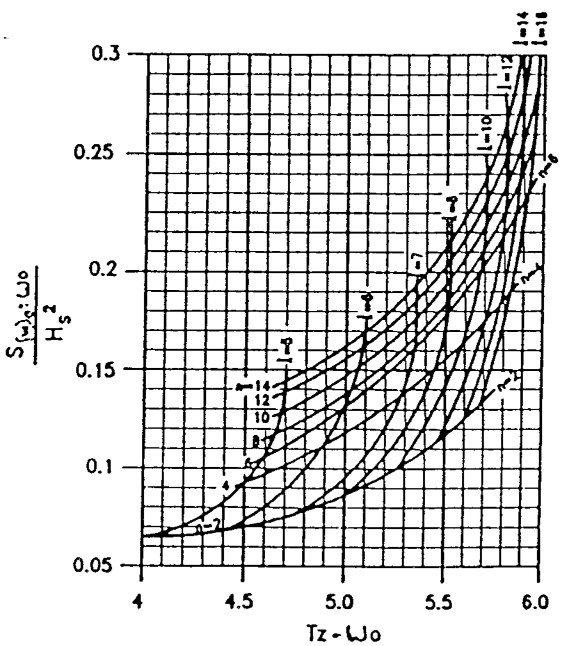

Fig. 6

4.4 Comparison of the 3- and 4-parameter spectra with a recorded North Sea wave spectrum

In the 1976 report on North Sea environment for the Norwegian Petroleum Directorate (see first reference [DNV]), a recorded North Sea spectrum was compared with the Pierson-Moskowitz and JONSWAP theoretical spectra (Figs. 2.4.4 and 2.4.5 of the report). The recorded spectrum had the following characteristics, as indicated:

H_s = 4.7 m
T_0 = 8.7 sec
T_z = 7.2 sec
$S_{(f)0}$ = 32.2 m^{2}s (measured from the figure, based on cyclic frequency f).

For these values, it then is determined:

$$\frac{\omega_0 \, S_{(\omega)_0}}{H_s^2} = \frac{S_{(f)_0}}{T_0 \, H_s^2} = \frac{32.2}{8.7\,(4.7)^2} = 0.1675$$

$$T_z \, \omega_0 = 7.2 \times \frac{2\pi}{8.7} = 5.20$$

The ℓ and n parameters then are determined for:

Ochi 3	from Fig. 5	ℓ = 13.2 with n = 4	equation (16)
Alt 3	from Fig. 5	ℓ = 8.35 with n = 7.35	equation (18)
4 Parameter	from Fig. 6	ℓ = 12.0 with n = 6.4	equation (11)

The equations are evaluated in terms of the cyclic frequency, f, and the resultant $S_{(f)}$ values are plotted through the range of frequencies in Fig. 7, along with the recorded wave spectrum. It is of interest to see that the Ochi 3 conforms most closely to the record through the high energy range, whereas the more complete 4-parameter shows the least agreement. A check made on the recorded spectrum

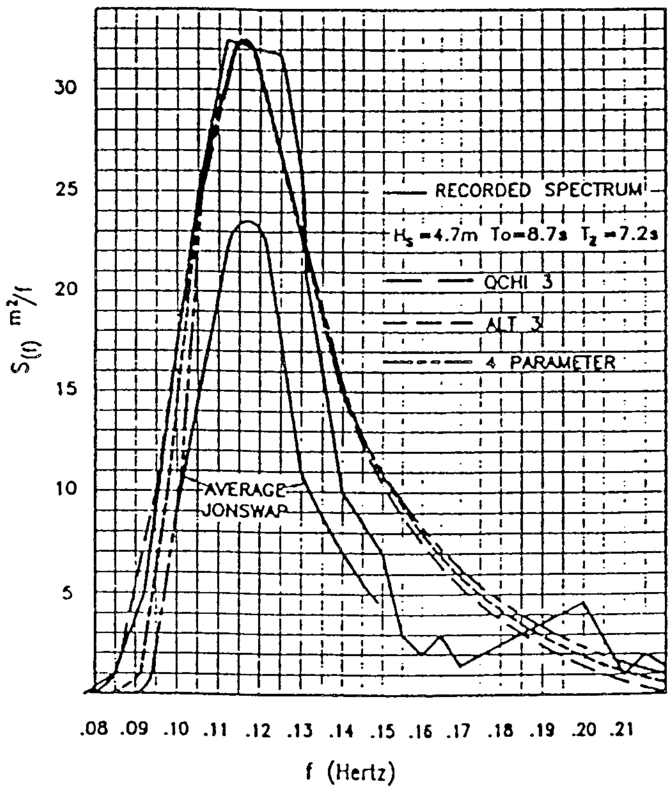

Fig. 7

through the full range of frequency shown on the figure indicates that the area under the curve is the expected $H_s^2/16$ = 1.38, whereas the zero upcrossing the period T_z = 7.8 seconds was somewhat greater than the reported 7.2 seconds (which reported period was used in the 4-parameter spectrum). Note that from equation (14) T_z = 8.0 sec for Ochi 3 and 7.6 sec for the Alt 3.

It is possible that there may have been an error in recording the wave data, but on frequent occasions, wave components of very high frequency (off the graph) and very low energy may exist, adding practically nothing to the overall energy content, but which may significantly increase the second moment to cause an unexpected decrease in T_z. Whatever the reason in this instance, it is apparent that the 3-parameter spectra represent the wave spectrum very satisfactorily without concern about the relative value of the zero upcrossing period.

This example may help to support a general conclusion that the 3-parameter formulations are more representative of sharply peaked spectra than are the two or four parameter; and the zero upcrossing period, T_z, has minimum significance in this application.

4.5 Analysis of a Gulf of Mexico hurricane spectrum

As a further example, the spectrum record of "Hurricane Georges" is analyzed with only the 3-parameter formulations.

From the observed data, the following characteristics are determined:

H_s = 10.8 m
f_0 = 0.08/sec
$S(f)_0$ = 180 m/s

from which $S(f)_0\, f_0/H_s^2$ = 0.123. Then, from Fig. 5:

Ochi 3	ℓ = 7.8	n = 4.0	equation (16)
Alt 3	ℓ = 6.5	n = 5.5	equation (18)

The equations are evaluated, and in this case are found to give results that are so nearly identical that one curve can be shown in Fig. 8 for comparison with the recorded spectrum. In effect, either Ochi 3 or Alt 3 could be used for further analysis purposes.

4.6 Six-parameter spectra

To represent double-peaked spectra, such as may occur when a fresh sea of high frequency is generated in the presence of an existing swell of low frequency, Ochi combined two of his three-parameter spectra. The details of the process and the resulting format are given explicitly in his paper, "Wave Statistics for the Design of Ships and Ocean Structures" (*Transactions,* SNAME, 1978) to which reference should be made.

Typically, the double-peaked spectra represent moderate sea conditions. When seas become more intense, the spectra approach the shape most represented by the single, sharply peaked 3-parameter formulation.

5. Wave height vs. wave amplitude

Traditionally, wave height has always been cited and specified, when describing the state of the sea, and for good reason. Data that were obtained, whether by visual estimate or by measured excursion of a float or other device, had no median point of reference to establish amplitudes up and down; only the height from crest to trough. Also, in the relatively simplistic days of analyzing regular waves only, height was the basic index even for trochoidal or Stokes waves, wherein differences between crest/trough amplitudes were distinct fractions of the wave height.

With this background, analyses of recorded wave data in the early days of spectrum theory concentrated on the height of crest to succeeding trough. Reasonably, these heights followed the Rayleigh distribution; and reasonably, their averages were close to the theoretical values. However, they were not close enough when compared to measurements of wave amplitudes about a mean surface—which became more readily determined as analytical processing improved, and which are the true variables on which the Rayleigh distribution is based (the details of which follow in the next section). Thus, the emphasis is on amplitudes—in the recommended

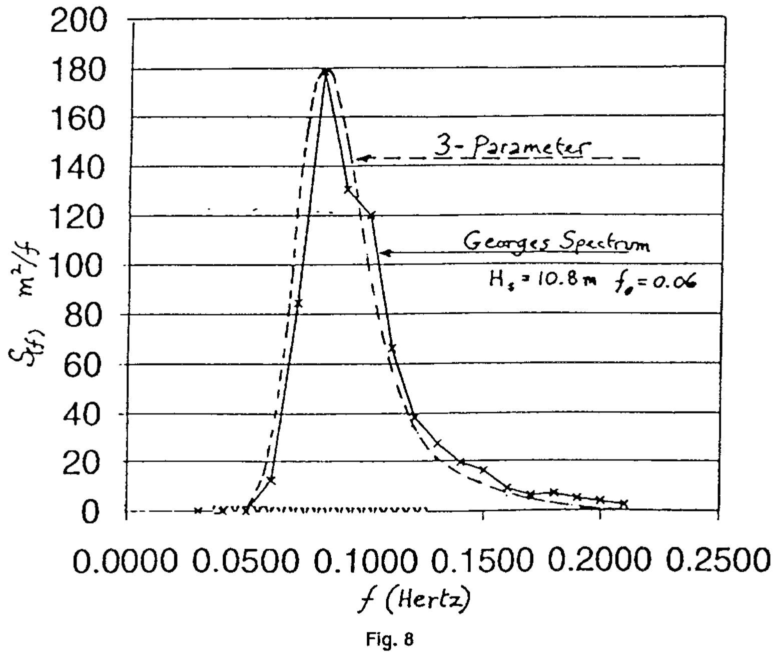

Fig. 8

spectrum formulations, the Rayleigh factors, and any hydrodynamic applications that would follow.

Nevertheless, wave heights (and periods) remain the highlighted reference, rather than amplitudes (and frequencies) in most treatises on the subject, including this one. It is a habit that is hard to break–and it need not be broken—as long as it is acknowledged that "wave height" in spectrum applications means "double amplitude" and not "crest to trough."

6. Factors of the Rayleigh distribution

6.1 The basic function and factors The Rayleigh distribution (Fig. 9) of wave height probability has, through the years, remained so confirmed in usage as to be considered a law of random sea behavior. In its clearest definition, it is described as follows:

The probability that the wave height is between H and $H + dH$ is

$$p(H)dH = \frac{2H}{\overline{H^2}} e^{-\frac{H^2}{\overline{H^2}}} dH \qquad (20)$$

where $p(H)$ is the percentage of occurrence per unit of height at the wave height H. It is also called the "probability density." $\overline{H^2}$ is the average of the square of the heights of all the waves in the spectrum $p(H)$.

It is convenient at this time to substitute $H_s^2/2$ for $\overline{H^2}$ in the above equation, and to note that

$$p(H)dH = \frac{4H}{H_s^2} e^{-\frac{2H^2}{H_s^2}} dH = e^{-\frac{2H^2}{H_s^2}} d\left(\frac{2H^2}{H_s^2}\right) \qquad (21)$$

Then, the probability that the wave height exceeds a certain value, H_i, is:

$$P(H>H_i) = \int_{H=H_i}^{\infty} e^{-\frac{2H^2}{H_s^2}} d\left(\frac{2H^2}{H_s^2}\right) = e^{-\frac{2H_1^2}{H_s^2}} \qquad (22)$$

It is seen that the probability of occurrence is higher for the lowest height (H_1) in the range from H_1 to ∞. Thus, if one is looking for the most probable highest wave in a total of N waves, the probability

$$\frac{1}{N} = e^{-\frac{2H^2}{H_s^2}}$$

leads to

$$\frac{H}{H_s} = \sqrt{\frac{1}{2} \ell n\, N} \qquad (23)$$

The percentile ranges of wave heights can be readily evaluated on this basis, using $1/N = 1 - \%$, where % is the decimal value of a stated percentile. Thus the 10% ranges are as given in the following listing:

10% of wave heights will be between 0.0 and 0.23 Hs
10% of wave heights will be between 0.23 and 0.33 Hs
10% of wave heights will be between 0.33 and 0.42 Hs
10% of wave heights will be between 0.42 and 0.50 Hs
10% of wave heights will be between 0.50 and 0.59 Hs
10% of wave heights will be between 0.59 and 0.68 Hs
10% of wave heights will be between 0.68 and 0.78 Hs
10% of wave heights will be between 0.78 and 0.90 Hs
10% of wave heights will be between 0.90 and 1.07 Hs
10% of wave heights will be over 1.07 Hs

Ranges for 5%, 1% or whatever can just as easily be calculated if one desires a closer division.

However, of greater practical importance is the determination of the heights of the highest waves. The 10% highest waves have been seen to be in the range above 1.07 Hs, but a more important index is that the average height of these waves is 1.27 Hs. To calculate the average or probable height of a certain number of highest waves, the probability of each wave height is multiplied by that wave height, and the full summation is divided by 1/N, the probability of the desired range. The average 1/N highest is then found from the equation:

$$\left(\frac{H}{H_s}\right)_a = \frac{N}{\sqrt{2}} \int_{\ell n\, N}^{\infty} \left(\frac{2H^2}{H_s^2}\right)^{1/2} e^{-\frac{2H^2}{H_s^2}} d\left(\frac{2H^2}{H_s^2}\right) \qquad (24)$$

For the average value of all the waves ($N = 1$, $\ell n\, N = 0$), the equation gives the definite integral $H/H_s = \Gamma(1.5)/\sqrt{2} = 0.63$. For all other values of N, the integral must be evaluated by stepwise computation. Figure 10 shows the computed values of the average highest up to $N = 1000$.

[Note that if equation (24) is written in terms of the original ($H^2/\overline{H^2}$) and is then evaluated for $N = 3$, the result gives $H^2 = 2\overline{H^2}$ (within 1/4%)—which is the value used for H_s^2 as the average for the 1/3 highest waves.]

One further relationship is of interest—the root mean square (RMS) of the 1/N highest waves. The equation for the mean square is:

$$\left(\frac{H}{H_s}\right)^2_{RMS} = \frac{N}{2} \int_{\ell n\, N}^{\infty} \left(\frac{2H^2}{H_s^2}\right) e^{-\frac{2H^2}{H_s^2}} d\left(\frac{2H^2}{H_s^2}\right) \qquad (25)$$

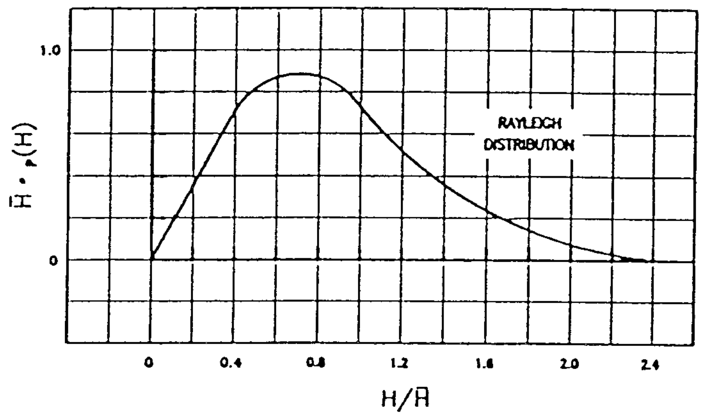

Fig. 9

N (NUMBER OF WAVES)	1	3	5	10	20	10^2	10^3	10^4	10^5	10^6
AVERAGE VALUE	0.63	1.00	1.12	1.27	1.40	1.67	1.97			
MOST PROBABLE $\sqrt{1/2\ \ell n\ N}$	0.00	0.74	0.90	1.07	1.22	1.52	1.86	2.15	2.40	2.63
R.M.S. $\sqrt{1/2\ (1 + \ell n\ N)}$	0.71	1.02	1.14	1.29	1.41	1.67	1.94	2.26	2.50	2.72

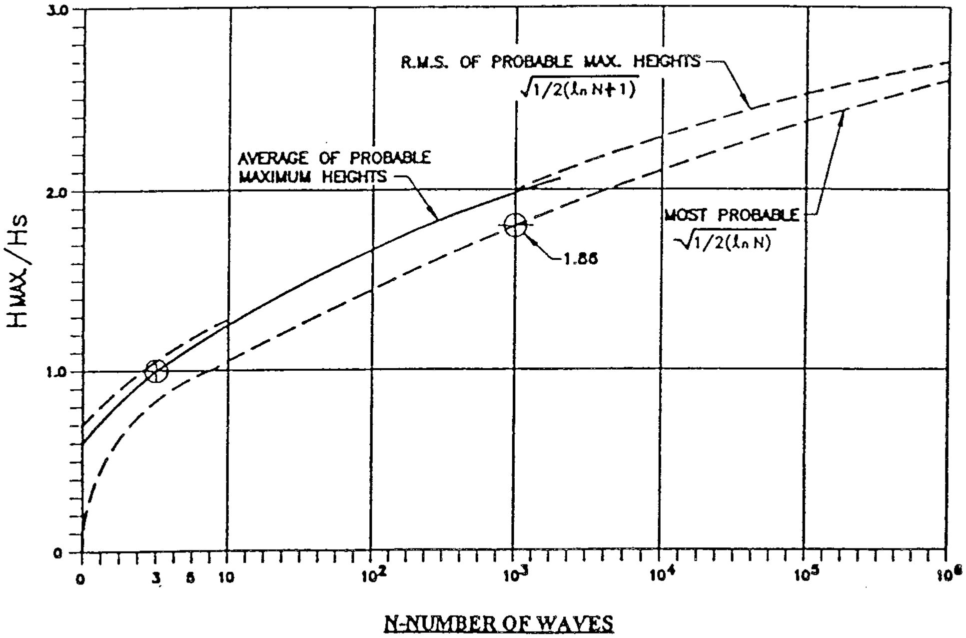

Fig. 10 Probable maximum wave height in N (number of waves) as a function of significant height, H_s

This is integrable, and results in the root mean square:

$$\left(\frac{H}{H_s}\right)_{\text{RMS}} = \sqrt{\frac{1}{2}(1 + \ell n\, N)} \tag{26}$$

The values of RMS are given in Fig. 10 along with those of the average highest and the most probable wave heights. It is seen that the RMS and the average are very close in value, to the extent that the RMS may be used as an approximation for the average where detailed calculations are not available.

The values of the most probable maximum height are significantly lower than the averages. The value of 1.86 as the most probable maximum in 1000 waves has long been considered the design criterion for offshore structures. If one considered the average value to be a more appropriate criterion, the wave height factor would be 1.97 for $N = 1000$, or if 1.86 were to be retained, it would be the average value for about $N = 400$.

In any event, the proper criterion for any specific site should ultimately be based on long range data for that area, including the effects of breaking waves (reducing the maximum heights and/or conversely increasing the local loading on the structure).

6.2 The broadness correction factor

The Rayleigh distribution is strictly valid if there is only one crest and one trough between each zero upcrossing point. That is, the number of waves considered in a given time, t, is t/T_z (where T_z is the average upcrossing period).

However, in every sea spectrum there are a number of additional crests and troughs present that do not cross the mean water level. Counting all crests, the number of waves considered would be t/T_c (where T_c is the average crest period). Then, due to all the additional waves of small height, the averages indicated by the Raleigh factors would have to be reduced to get the true average 1/3 height, 1/10 height, etc. for all of the waves in the record.

This reduction factor is shown to be

$$\left(1 - \frac{\varepsilon^2}{2}\right)^{1/2},$$

where ε is the broadness factor:

$$\varepsilon = \left(1 - \frac{m_2{}^2}{m_c m_4}\right)^{1/2}$$

or with

$$T_z = 2\pi\left(\frac{m_c}{m_2}\right)^{1/2} \text{ and } T_c = 2\pi\left(\frac{m_2}{m_4}\right)^{1/2},$$

the reduction factor becomes

$$\left[\frac{1}{2} + \frac{1}{2}\left(\frac{T_c}{T_z}\right)^2\right]^{1/2}.$$

Thus, if one needs to consider every crest in the record, the Rayleigh factors for the 1/N highest waves (now counting every crest as a wave) would be multiplied by the previous reduction factor to get the true average heights of those 1/N highest waves.

However, for most design purposes, it is sufficient to consider only the maximum elevation within one "wavelength" (between zero upcrossings) as is represented by the standard Rayleigh function, neglecting whatever lessening effect any secondary crests may have on the "average values." If average maxima are specified in any particular design evaluation, a measure of conservatism is provided in the use of the standard factors.

Furthermore, as Ochi has shown, the most probable maximum height, as determined for t/T_z number of waves, remains unchanged regardless of the broadness factor.

Reiterating: if one establishes the average zero-upcrossing period as the proper index for the number of waves (which is the most common practice) the Rayleigh factors can be used directly for design purposes, without correction.

7. Limiting heights of breaking waves

Breaking waves are in evidence in all seaways of any consequence. In some instances, the breaking action takes the top off the higher waves in the spectrum. In others, it may simply wipe out some of the low amplitude components in the high frequency range, resulting in the more sharply peaked spectra characteristics of North Sea storms. In any event, the breaking phenomenon dissipates wave energy, and can alter the wave probability relationships.

Reports on model tests of irregular waves indicate that breaking occurs when $H = 0.02gT^2$, which corresponds to a steepness ratio of $H/L = 1/9$ by Stokes' third approximation. This is in keeping with breaking of regular waves, which range in steepness from the theoretical 1/7 to about 1/10 in model tests. For analytical purposes, the relationship $H = cT^2$ is used in developing the general breaking wave probabilities; specific values of c may then be assigned for steepness ratios of interest.

The probability distribution of T^2 is taken to be of the Rayleigh form similar to that of the wave height, and the two variables are considered to be independent of each other (no correlation effects). It may be noted that Bretschneider used this premise in developing the standard spectrum function given in Section 2, and while some recent studies show variations in period distributions and correlation effects, the influence on breaking probabilities is considered small. The following relationship is then utilized:

$$p\,(T^2)\,dT^2 = \frac{2T^2}{T_z^4}\,e^{-}\left[\frac{T^4}{T_z^4}\right]dT$$

which states the probability that the square of the period lies between T^2 and $T^2 + dT^2$.

From the general Rayleigh distribution of wave height, the probability of exceedance is

$$e^{\frac{-2H^2}{H_s^2}},$$

for which the breaking value

$$e^{\frac{-2c^2T^4}{H_s^2}}$$

may be substituted. Applying to this the full range of T^2 probabilities, the total probability of breaking waves in a given seaway is:

$$P\,(H \geq cT^2) = \int_0^\infty e^{-\frac{2c^2T^4}{H_s^2}}\left(2\frac{T^2}{T_z^4}\,e^{-\frac{T^4}{T_z^4}}\,dT^2\right) \qquad (27)$$

or simplifying:

$$= \int_0^\infty e^{-\frac{T^4}{T_z^4}\left(\frac{H_s^2+2c^2T_z^4}{H_s^2}\right)}\,d\left(\frac{T^4}{T_z^4}\right)$$

Let $\kappa = \frac{T^4}{T_z^4}\left(\frac{H_s^2+2c^2T_z^4}{H_s^2}\right)$ and $d\chi = \frac{2T^2}{T_z^4}\left(\frac{H_s^2+2c^2T_z^4}{H_s^2}\right)$

Then

$$P\,(H \geq CT^2) = \frac{H_s^2}{H_s^2+2c^2T_z^4}\int_0^\infty e^{-\chi}dz = \frac{H_s^2}{H_s^2+2c^2T_z^4} \qquad (28)$$

The probability of breaking for waves higher than a designated value Hl is then:

$$P\,(Hl \geq cT^2) = \frac{H_s^2}{H_s^2+2c^2T_z^4}\int_{T^2=Hl/c}^\infty e^{-\chi}\,d\chi \qquad (29)$$

$$= \frac{H_s^2}{H_s^2+2c^2T_z^4}\,e^{-\frac{Hl^2}{C^2T_z^4}\left(\frac{H_s^2+2c^2T_z^4}{H_s^2}\right)}$$

$$= \frac{H_s^2}{H_s^2+2c^2T_z^4}\,e^{-\frac{2Hl^2}{H_s^2}\left(\frac{H_s^2+2c^2T_z^4}{2c^2T_z^4}\right)}$$

Substituting the parameter $X = H_s^2/2c^2T_z^4$, the final expression is

$$P\,(Hl \geq cT^2) = \frac{X}{1+X}\,e^{-\frac{2Hl^2}{H_s^2}(1+X)} \qquad (30)$$

Thus, it is seen that the breaking probability of waves higher than Hl has a Rayleigh distribution applied to the probable total of breaking waves in the spectrum.

From equation (30), the highest breaking waves can be determined.

Most probable highest breaking wave in N *number of waves*

The probability of exceedance

$$= \frac{1}{N} = \frac{X}{1+X}\,e^{-\frac{2Hl^2}{H_s^2}(1+X)}$$

Thus

$$\frac{Hl}{H_s} = \left[\frac{1}{2\,(1+X)}\,\ell n\,\frac{NX}{1+X}\right]^{1/2} \qquad (31)$$

Average highest breaking wave in N *number of waves*

$$\frac{H_{0_e}}{N} = \int_{T^2=H/C}^\infty H\,\frac{X}{1+X}\,e^{-\varkappa}d_\varkappa$$

which arranges to

$$\frac{H_a}{H_z} = \frac{NX}{\sqrt{2}\,(1+X)^{3/2}}\int_{\ell n\frac{NX}{1+X}}^\infty \varkappa^{1/2}\,e^{-\varkappa}\,d_\varkappa \qquad (32)$$

with $\varkappa$ put in the form of

$$\frac{2H^2}{H_s^2}\,(1+X).$$

The equation is not integrable, and must be evaluated by computation.

RMS highest breaking wave in N *number of waves*

$$\frac{H^2_{\text{RMS}}}{N} = \int_{T^2=H/C}^\infty H^2\,\frac{X}{1+X}\,e^{-\varkappa}\,d_\varkappa$$

which arranges to

$$\frac{H^2_{\text{RMS}}}{H_s^2} = \frac{NX}{2\,(1+X)^2}\int_{\ell n\frac{NX}{1+X}}^\infty \varkappa\,e^{-\varkappa}\,d\chi.$$

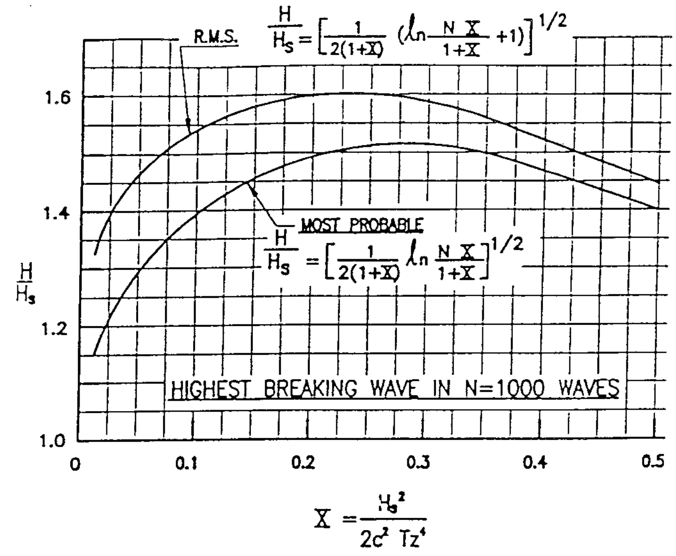

Fig. 11 Note: As in the case of the non-breaking wave spectrum, given in Section 6, whether one considers the most probable highest breaking wave or the average highest (close to the RMS value) to be the more appropriate criterion, the ultimate determination should be based on long-range data for a given specific site

This is integrable and leads to the value

$$\frac{H_{\mathrm{RMS}}^2}{H_s} = \left[\frac{1}{2\,(1+X)} \left(\ell n \, \frac{NX}{1+X} + 1 \right) \right]^{1/2} \qquad (33)$$

As with the basic factors of the Rayleigh distribution, the value of the average highest wave is almost identical to that of the root mean square (RMS) which is more readily determined for all values of X and N. Figure 11 shows the range of the most probable highest breaking wave and the corresponding RMS value, for $N = 1000$ waves. For other values of N, the breaking wave heights can be readily determined from the formulations.

8. Wave generation as a function of wind and fetch

It has long been noted that waves at sea increase in intensity, both as to height and significant period, with increased wind speed and the distance over which the wind acts. Through the years, a large number of observations have been recorded that show the general trend, and formulations have been developed for application, most notable by Bretschneider.

Modifying his equations slightly, to account for more recent indications of wave generation limits, the following is proposed, in nondimensional form:

$$\frac{gH_s}{U^2} = 0.243 \tanh \left[0.011 \left(\frac{gF}{U^2} \right)^{1/2} \right] \qquad (34)$$

$$\frac{gT_s}{U} = 7.54 \tanh \left[0.077 \left(\frac{gF}{U^2} \right)^{1/4} \right] \qquad (35)$$

where

H_s = significant wave height
T_s = significant wave period
U = wind speed
F = fetch, distance over which wind acts

The values of the above equations are plotted on the complied data, as given by Wiegel in Fig. 12, and are seen to conform satisfactorily throughout the range. It may be noted also that a reasonable fit to the data for wave velocity is identical in form to that for the wave period, differing only in the ordinate whereby $C/U = 0.12\,(gT_s/U)$. Considering that the zero-upcrossing period may be presented as $T_z = T_s/\pi^{1/4}$, the wave velocity is seen to be:

$$C = 0.12\pi^{1/4}\,(gT_z) \approx \frac{gT_z}{2\pi} \qquad (36)$$

For evaluation, the foregoing equations for wave height and period are put in the form:

$$H_s = V_k^{\,2} \times 0.022 \tanh \left[2.88 \left(\frac{N}{V_k^{\,2}} \right)^{1/2} \right] \qquad (37)$$

$$T_z = V_k \times 0.298 \tanh \left[1.25 \left(\frac{N}{V_k^{\,2}} \right)^{1/4} \right] \qquad (38)$$

where

H_s = significant wave height, ft
T_z = zero-upcrossing period, sec
V_k = wind speed, knots
N = fetch distance, nautical miles

Waves of maximum steepness (height to period ratio) are seen to be generated by high winds over a short period of time. Thus for the condition where $N/V_k^{\,2} \to 0$, *the value* $T_z = 8.4\,\sqrt{H_s/g}$ is determined from the above (Bretschneider's early value was $8.9 \approx 9.0$).

Yet further evaluation of the above equation indicates that the maximum steepness occurs at about $T_z = 8.0\,\sqrt{H_s/g}$ for any wind speed, but only after the wind has been progressing forward for about 1 or 2 hours. Some of today's authorities are specifying this value as the characteristic of the "maximum storm" to which marine structures may be subjected.

Thereafter, as the wind proceeds further, the wave steepness decreases even though the height and period continue to increase to some extent, until finally the "fully developed sea" condition is achieved. This is the point where generation stops, and no further increase in wind duration or fetch will change the wave characteristics.

This ultimate condition is evaluated, taking $N/V_k^{\,2} \to \infty$, and it is seen that

$$T_z = 11.4\,\sqrt{H_s/g} \approx 0.3V_k$$

This compares favorably with Pierson-Moskowitz, whose formulations reduce to

$$T_z = 11.1\,\sqrt{H/g} \approx 0.27V_k$$

[Note that P-M measured wind 60 ft above the water surface, whereas other data were taken at 30 ft or 10 m.]

Figure 13 shows the interrelationship of wind, fetch, wave height, and period through the range of practical application, as evaluated from the equations. It is of special interest to note that the rate of increasing wave height with continuing wind begins noticeably diminishing at the point where the wave speed ($gT_z/2\pi$) has reached the wind speed; and thereafter there is little height increase up to the point of the fully developed sea. [That there should be any height increase at all beyond the condition of equal speeds is due to the fact that some wave components are still slower than that of the average and the wind continues to impart energy, albeit in ever decreasing amounts until total equilibrium is reached.]

This is shown again in Fig. 14, where the limiting conditions of wave generation are superimposed on a wave scatter diagram typical of the upper North Sea off the coast of Norway. It is seen that most of the waves in this area have the characteristics of fully developed seas and beyond that, into the wave decay zone. Understandably, there is practically unlimited fetch with storms generating in the North Atlantic to the west, and as they abate in approaching the North Sea,

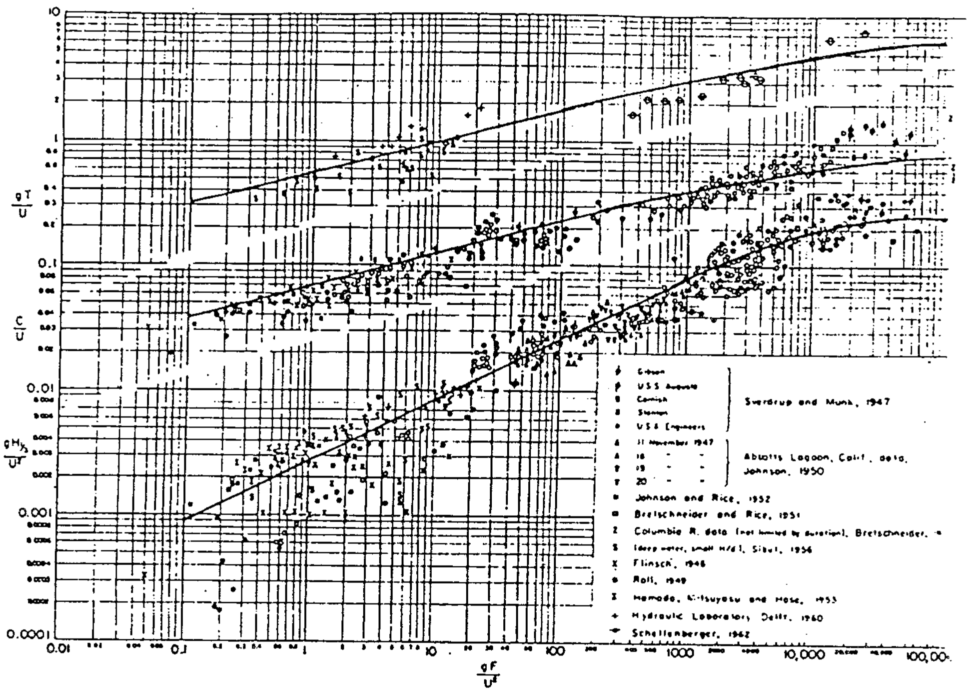

Fig. 12 Relationships among fetch, wind velocity, and wave height, period, and velocity (after Wiegel, 1961)

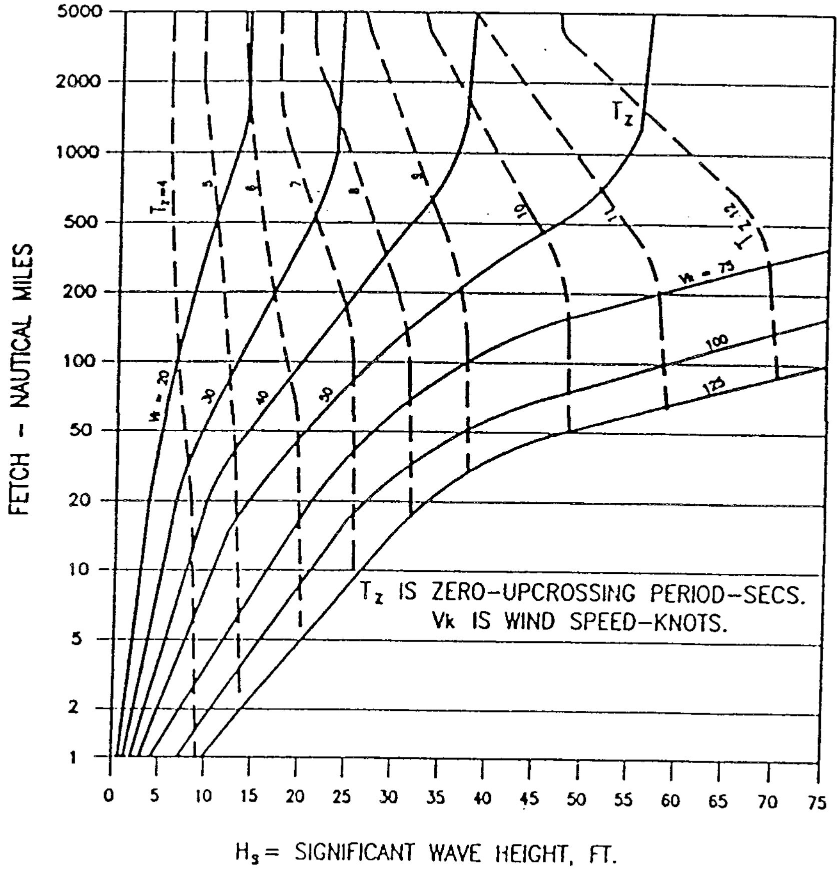

Fig. 13 Wind-generated wave relationships

the low-period waves are filtered out and the resulting seas in the area are of longer period and somewhat reduced height than those that were initially generated.

In contrast, the wave characteristics in the lower North Sea between Scotland and Denmark are seen to be well within the generating range for a good part. Here the fetch is definitely limited and the wind speed somewhat less (except for the infrequent storm that blows down from the north), so

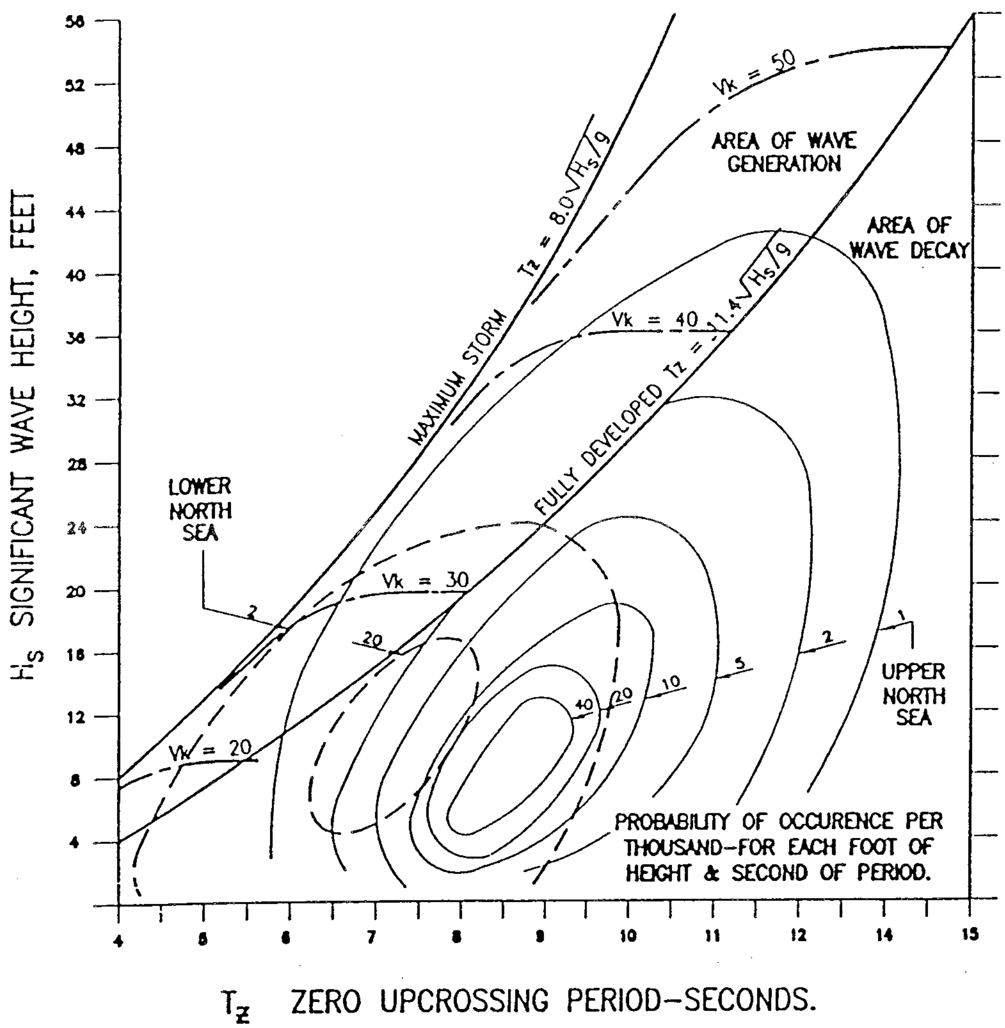

Fig. 14 Typical North Sea wave scatter diagram (with wave-generating functions superimposed)

that waves are necessarily generated within the area and with lower periods (characteristically, wave steepness is greater here than most anywhere else—a long noted observation).

Considering the much greater number of waves that are indicated to be in the decay zone than in the generating area, as indicated in Fig. 14, it would be reasonable to question the limit given by Pierson-Moskowitz. There is sufficient scatter in the data shown in Fig. 12 to justify a different selection of values for the fully developed sea that would extend the area of wave generation to encompass more of the observed wave scatter records. This may not be important for areas such as the North Sea where detailed wave information continues to be compiled and where there is little need to correlate this data with a formulation that has no further purpose. However, in virgin areas, it may be useful to consider an extension of the range of wave generation for estimating purposes.

At the other extreme, significant wave heights of large magnitude in association with the maximum storm limit, as shown, are well beyond the North Sea data, as expected. They are generally only approached on rare occasions—in the North Atlantic with high sustained winds, and in the hurricane-prone waters off the Americas. The limit, itself, is rather extreme insofar as it indicates that the average significant wave steepness ($2\pi\ H_s/gT_z^2$) is about 1/10 in the range where breaking (or breaking down) is to be expected, particularly where those waves higher than the significant are considered.

In any event, it is recognized that over most ocean areas, the wind is not unidirectional for any long period of time nor does it progress at a steady rate of speed or free of gusts. In addition, it is difficult to confirm fetch distances over which it blows. Further, there is no indication from the data as to the shape of the sea spectrum that one might expect as a function of the determined significant wave height and period; whether it be the standard two-parameter spectrum, a steeper three-parameter, a Jonswap or whatever, as is necessary to assign in any detailed analysis of a marine structure in that environment.

With the great advances in meteorology and accumulation of sea data (by satellite and direct measurement), we should anticipate more detailed and precise information to be developed by oceanographers in the near future, both as to hindcasts (what to expect, generally, in any given area) and to forecasts (what to expect from an oncoming storm). In the meantime, the relationships given herein should serve as a "state of the art" indication of how wind speed and duration determine the particulars of the sea. They are presented on this basis.

References and recommendations

It will have been noted that there are a minimum number of references indicated in the text (two). There would be far too many if all pertinent contributions to the subject (over the last 40-plus years) were credited. Instead, each of those two given references contains a significant number of references of their own that generally encompass the developments leading towards the present. Furthermore, they both extend into the realms of long range predictions and other manifestations of sea characteristics that ultimately need to be addressed. A third reference which is listed here, provides an overview of the present state of the art concerning all those same aspects, with further references to more recent works.

All three are excellent for reference and for their presentations of advanced sea technology, increasingly important to those who wish to probe more deeply into this field.

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Appendix

A sea spectrum primer

Makeup of the sea

The sea is never regular. It does not take the form of a series of uniform waves, of constant height and length, proceeding in a steady and reliable sequence. Rather, a true sea is a random phenomenon, a myriad of waves of all different sizes, lengths, and directions, jumbled together in an apparent confusion, as a result of wind-generated (usually) disturbances of different intensities, locations, and directions.

The theories which are now well substantiated, explain how such seas build up, change shape, disperse, and so forth. While we cannot (as yet) make precise long range predictions of exactly when and where storms of what intensities will arise, we do have fairly good hindsight information on whether they will arise in a general area and how frequently. Further, once a sea has been generated, we have a good idea of its characteristics, how it will behave, and how it will affect bodies in its path.

First, consider the basic two-dimensional seaway—that which is classed as an "irregular sea," as generated by a broad-scoped wind. The wave crests are continuous in a breadthwise direction, and all waves move in the same forward direction. Ultimately, of course, as an actual sea moves out of the storm area it spreads out sideways (losing height as it does), and thins out progressively, as the longer waves in the sea tend to outrun the others. If in the course of its travels, it also meets waves from other disturbances coming from different directions, as is usually the case, the shortcrested "confused sea" results. While this three-dimensional confused sea is more prevalent in nature and is of importance in such instances as evaluating a long range history of the motions of a ship, the two-dimensional irregular sea is considered to have maximal effect on a body situated within it, particularly at or near the area where the sea has been generated.

As to the actual make-up of the seaway (in two dimensions now) the mathematician may describe it as:

> "An infinite number of uni-directional sinusoidal waves, with continuous variation in frequency; with each wave of an infinitesimal height and random in phase."

For more immediate understanding and visualization of the seaway, we may more simply describe it as:

> "A collection of a great number of simple, regular waves of different lengths, all of small height and all mixed together with no apparent relation to each other except that they are all there and are all traveling in the same direction."

The result is an irregular sea, with no set pattern to the wave height, length or period. For illustration, we may consider the result of combining a small number of uniform waves, of different lengths and heights (Fig. 15). It can be seen how irregular the resulting wave is, as formed by only four basic regular waves, and it requires very little imagination to foresee that with an infinite number of simple waves, all of different lengths (or periods) the summed-up resultant wave would be completely irregular and impossible to predict in shape. In fact, the most distinctive feature of the irregular sea is that it never repeats its pattern, from one interval to any other. Thus, we cannot characterize or define an irregular sea by its pattern or shape.

There is, however, one way we can define the sea in simple terms. Its total energy must necessarily be made up of the sum of the energies of all the small, regular waves that make up the sea—no more and no less.

Note that the energy of a simple, sinusoidal wave is readily shown to be $\rho g H^2/8$, for each square foot of water surface (ρg is the unit weight of water, H is wave height from crest to trough). Then the total energy in each and every square foot of the seaway is

$$\text{Energy} = \frac{\rho g}{8}\left(h_1^{\,2} + h_2^{\,2} + h_3^{\,3} + \ldots\ldots.\right)$$

or simply a constant times the sum of the squares of the heights of all the simple, small waves that exist in that seaway.

Thus, the intensity of the sea is characterized by its total energy, and what is most important, we can show the individual contribution made by each of its component waves. In other words, with each component wave of different length or period, or more conveniently, of different frequency, we can show how the total energy of the sea is distributed according to the frequencies of the various wave components.

The distribution is what we call the energy spectrum of the sea, or more simply the "sea spectrum." For simple illustration, let us devise a crude spectrum made up of those same four regular waves used previously. It would look something like Fig. 16.

The ordinate of the "curve" is expressed as *energy-seconds,* and may be regarded as an abstract term conveniently selected so that the area under the spectrum curve represents the entire energy of the system, when plotted on a frequency base that has the dimension of *1/seconds.* Note that while we have centered the energy of each component wave at its designated frequency, we have given it a "bandwidth," so that the *energy-seconds* ordinates have finite values and so that the curve has a semblance of continuity over a wide range of frequency.

And continuity it must have, according to observed behavior of the actual sea. That is, we must consider that the sea contains a great number of waves, that vary slightly in frequency, one from the next. Otherwise, if we only had four different waves, as per this elemental example, or even ten or twenty, sooner or later we would see the wave pattern of the sea repeating itself exactly. Furthermore, as the sea proceeded into new areas, it would separate into groups of regular waves, followed abruptly by areas of calm, and so on (since waves of shorter frequency and greater length are faster and soon outrun those of longer frequency and shorter length). These conditions are

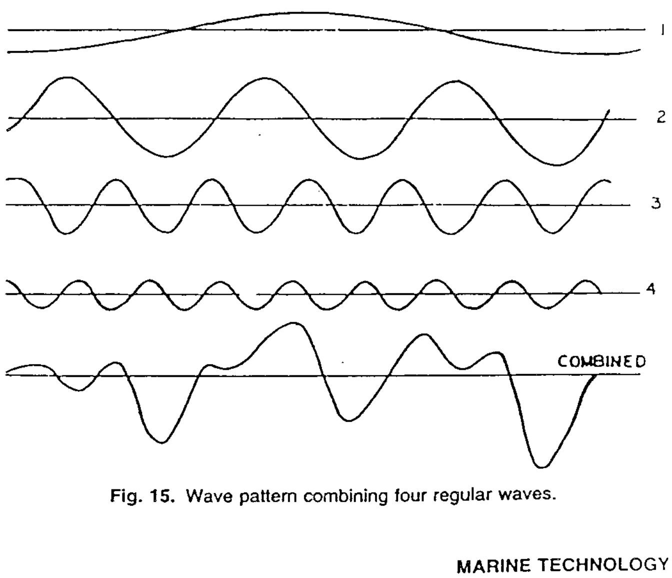

Fig. 15. Wave pattern combining four regular waves.

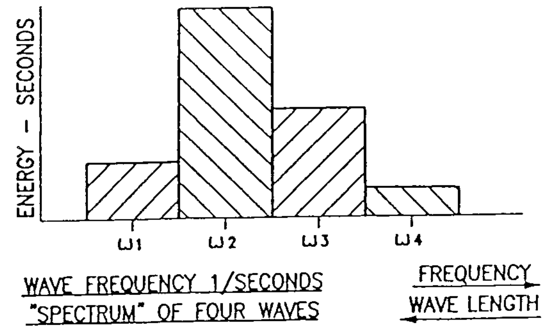

Fig. 16

certainly not representative of those that actually exist, since the sea disperses or "dissipates" in a continuous and gradual manner.

So we need to consider that a sea is composed of a very great number of different frequency waves; and for a given amount of total energy of the sea, we can see that the greater the number of waves considered, the less energy (or height) each of these component waves possesses. Ultimately, then, the most factual energy spectrum of the sea is a smooth, continuous curve made up of the contribution of an infinite number of regular waves, all of different periods of exceedingly small height. (See Fig. 17).

Add to this, the fact that we do not know, and cannot predict, what relative position one component wave has to the next (i.e., what the phase relationship of one sine wave has to another), we can never tell just when a number of waves will group together to form a high sea wave or when they will tend to cancel out, or whatever, in any systematic sequency. Thus, we have "randomness," and we now see the logic of the mathematician's definition of the seaway.

We may note here that while we have been using *energy-seconds* as the ordinate of the curve, resulting in *energy* as the area, we may conveniently substitute *feet*^{2}*-seconds* for the ordinate, and ft^2 for the area, as a direct indication of component wave-height variations, since energy and height^2 are directly proportional.

We shall see later just what values may be ascribed to the spectrum curve; that is, what mathematical function can be used to define both the shape and area under the curve for given conditions of wind force or sea state. For now, we may simply note that the spectrum builds from the high-frequency end. That is, for a given wind-speed, the first waves generated are those that are of short length; and then, as the wind continues to blow, longer and longer waves are generated until finally the condition known as "the fully developed sea" is reached, where the system is stable and no further effect is

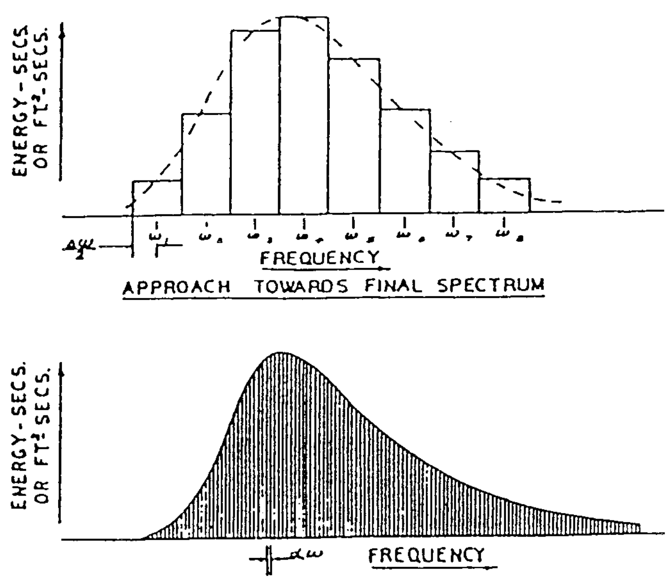

Fig. 17 Final spectrum

produced, regardless of how much longer the wind blows and over how much more area.

Thus, as more and more energy is put into the seaway, its spectrum changes. As it grows (Fig. 18), its total scope includes more low frequency (longer) waves, and its maximum value shifts towards the low frequency end, as well. This is true also for fully developed seas, as a function of increasing wind speed.

Heights of the sea waves

Now that the form and content of the spectrum becomes clear, the question remains how to determine the actual physical wave heights that are produced within the seaway itself. As stated previously, we cannot predict the actual pattern of the sea surface insofar as what wave follows what other wave. However, we can predict by statistical methods how often waves of various heights will occur over any given period of time, for a sea of a given amount of energy.

The most acceptable formulation to date for deriving statistics of wave-height distribution is one which has been corroborated by actual wave measurements that have shown a very consistent pattern over many years of investigation. To illustrate how such a distribution is determined, the heights of all the waves in a given record are measured and the percentage of occurrence calculated; that is, the *number* of waves under 2 ft high, from 2 to 4 ft high, 4 to 6 and so forth, are each divided by the *total number* of waves in the record. Then these percentages are plotted against the wave heights themselves, resulting in the figure known as a "histogram" (Fig. 19).

It was found that one simple form of curve fits most sea-wave histogram records very closely. This is known as the "Rayleigh" distribution, and is written

$$p(H_1) = \frac{2H_1}{\overline{H^2}} e^{-\frac{H_1^2}{\overline{H_z^2}}}$$

This may be expressed as "the percentage of times that a wave of height, H_1 ft (plus or minus 1/2 ft) will occur in all the waves of that series."

Let us identify the term $\overline{H^2}$. This is the average of all the squared values of the wave heights in the record, or expressed mathematically:

$$\overline{H^2} = \frac{1}{N}\sum_{l=1}^{l=N} H_l^2$$

where N is the total number of waves in the record.

Intuitively, we can see that this value, being the average over the entire area of the sea (for the actual sea waves, which are made up by the collective action of the small regular waves in the spectrum) should be very close in representing the average energy of the sea; that is:

$$\frac{\rho g}{8}\overline{H^2} \approx \frac{\rho g}{8}(h_1^2 + h_2^2 + h_3^2 + ...) = Energy$$

Thus, once we know the area under the spectrum curve, we can relate it directly to the Rayleigh distribution formula, and determine from this all sorts of useful probabilities of occurrence of different wave heights. For example:

$$P(H > H_l) = 1 - \int_o^{H_l} \frac{2H_l}{\overline{H^2}} e^{-\frac{2H_l}{\overline{H^2}}} dH = e^{-\frac{H_l^2}{\overline{H^2}}}$$

gives "the probability that the wave height will be greater than H_l," or in other words, out of a number of waves, N, there will be $Ne - H_l^2/\overline{H^2}$ waves that will be higher than H_l.

We can go from this into determining what the average wave height will be, or the average height of the 1/3 highest wave, or the 1/10 highest wave, and so forth. For instance:

Average wave height, $H_o = 0.89\sqrt{\overline{H^2}}$

Average height on 1/3 highest waves, $H_{1/3} = 1.41\sqrt{\overline{H^2}}$

Average height of 1/10 highest waves, $H_{1/10} = 1.80\sqrt{\overline{H^2}}$

A more comprehensive tabulation of these various statistical prob-

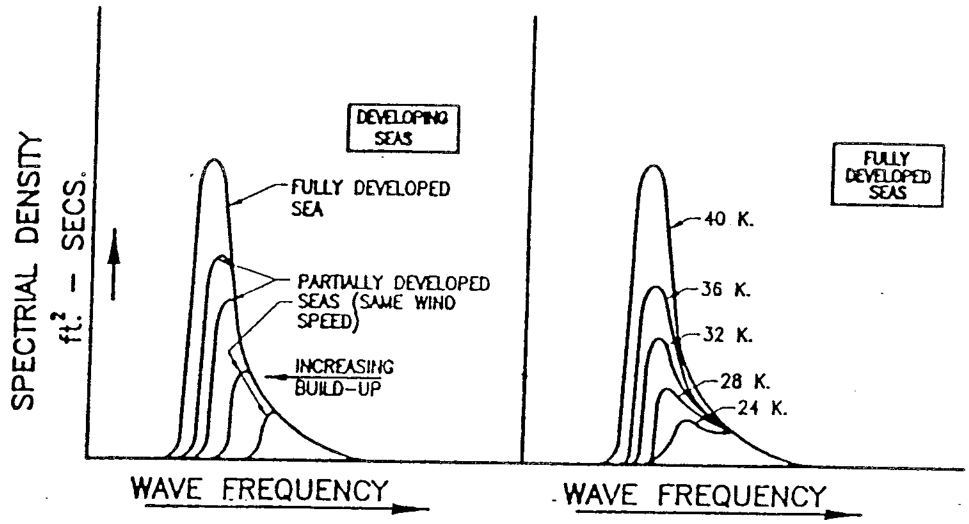

Fig. 18 Growth of spectrum

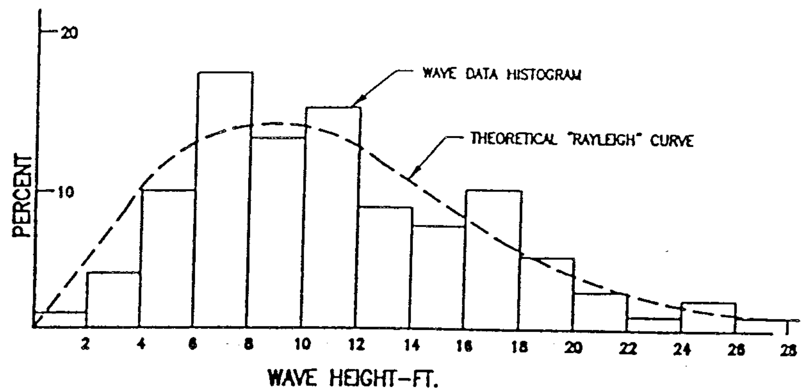

Fig. 19 Histogram of wave-height measurements

abilities, all derived from the Rayleigh distribution, are given in Section 6.

Let us note here the particular importance of "the average height of the one-third highest waves." This is identical to the value assigned to the "significant height," which stems from the fact that psychologically (and physiologically) an observer tends to neglect the small waves and only notices the large ones when evaluating the wave conditions he is experiencing. The *significant height* remains a most significant index, since the practice of reporting sea conditions on this basis is widespread, among oceanographers and seafarers alike.

Let us dwell briefly on the validity of the Rayleigh distribution, in the sense of whether and how it fits in the overall statistical theory which has been developed for sea waves. In the first place, the laws of statistics indicate that the sea surface should follow the well-known "Gaussian" or "normal" distribution (the good old "bell-shaped" curve)—that is, the probability that the water surface at a given location has a certain elevation could be determined using the normal distribution.

However, this in itself does not do us much good, since we are interested primarily in the frequency and value of the maximum (or minimum) elevations, that is, crests and troughs. A more useful formulation would then be the envelope curve of the maximum surface elevations, and happily this is what the Rayleigh distribution works out to be.

There is a theoretical reservation. The Rayleigh distribution is mathematically indicated to apply accurately only to a narrow spectrum (one which is highly peaked in shape, and most of the energy contained in a narrow range of frequency), or to a narrow band of a general spectrum. It presumably loses a validity when applied to an entire broad spectrum, or a multi-peaked spectrum, unless certain corrective factors are applied.

Furthermore, mathematical laws are obeyed only if we consider the amplitude of the wave, measured either above or below the still water level. Therefore, theory says we cannot precisely deduce wave heights (measured from crest to succeeding trough) from a given distribution, nor can we derive the proper distribution from measurements of wave height—all simply because the sea does not give us individual waves whose crests are the same distance above the still water level as the succeeding troughs are below. In other words, the probability of a crest being a certain height above still water level is not associated with an equal probability that the succeeding trough will be the same distance below.

Where does all of this leave us? Well, repeated tests and observations indicate that the uncorrected Rayleigh distribution still gives excellent correlation, regardless of spectrum shape, and does so with sufficient accuracy for engineering application, and no other distribution has yet been shown to give as consistently good results.

Application to ship behavior

Now, with the knowledge that the Rayleigh distribution holds for the wave spectrum, regardless of its shape, and that we can thereby determine wave heights in the seaway and their probability of occurrence, can we now take the bold, giant step forward and apply this same analysis towards other things that occur in that sea? Can we, for example, derive a "heave" spectrum for a ship in that sea, whereby instead of plotting the square of the height of each of the component regular waves, we plot the square of the heave that a regular wave of that height and frequency would produce? Can we do it for other ship motions, and for wave forces on a body—and do those same statistical factors apply to the resulting spectrum, for determining the magnitudes of the motions or forces, and their probabilities of occurrence in that seaway?

Fortuitously, the answer is a firm yes, as has been substantiated by a goodly number of performance tests and analyses. And this is the real import of the theory and the developed techniques—for the wave data would have little significance in itself, if we could not reliably determine the forces and motions of the bodies in those waves.

Now then, we need to devise the spectrum for the particular motion or force on the body. For this we need:

(*a*) the height characteristics of the component waves of different frequencies that occur in the sea. These are of course given by the sea spectrum, in terms of ft^2 - *secs*.

(*b*) the unit response of the vessel for each of the component waves of different frequency. For example, if we are investigating motions such as pitching, then we need to determine the maximum pitch angle (up or down) for a wave 1 ft high, and we must do this for a sufficient number of regular waves of selected frequencies (different length), corresponding to the range of frequencies given by the sea spectrum. If we are looking for wave forces on the body, we determine the maximum force (plus or minus) per foot of wave height, for a similar range. The ordinate of the unit response is then pitch angle/foot of height, or force/foot of height, or whatever it is we are investigating.

Then, by multiplying the spectral wave height by the *square* of the unit response (at the same frequency) we get ordinates of $(pitch\ angle)^2$-*secs.*, or $force^2$-*sec.*, etc., and plotting these for the full range of frequencies involved, we get the particular motion or force spectrum we desire.

Then we can proceed as for the wave spectrum itself, by getting the area under this new spectrum curve and applying the same Rayleigh constants to get one-third highest motion, (or force) one-tenth, etc., as we may desire.